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A realization of the dynamical group for the square-well potential and its coherent states

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Abstract
A realization of the ladder operators for the infinitely deep well potential is presented. Both the asymmetric and symmetric wells are discussed. It is shown that these operators satisfy the commutation relations for the $SU(1,1)$ group. Closed analytical expressions for the matrix elements of some relevant functions are obtained. Perelomov coherent states for this system are explicitly constructed. The behaviour of the dispersion relation and the time evolution for such states are examined.

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1. Introduction

The square-well potential is analysed in quantum mechanical textbooks, both as an introduction to fundamental concepts and to exemplify basic mathematical techniques. For the case of an infinite potential, the solutions of the Schrödinger equation can be found explicitly in one or more dimensions [1]. On the other hand, exactly solvable quantum systems, such as the n-dimensional harmonic oscillator, the one-dimensional Morse oscillator and the non-relativistic hydrogen atom, owe their analytic solution to the presence of a symmetry or dynamical group and hence lend themselves to an algebraic treatment [2]. It is thus interesting to investigate whether the square-potential solutions are related to a symmetry and whether an algebraic treatment is feasible. This problem was considered in [3, 4], where it was concluded that the dynamical group for this system is $SU(1,1)$, while in [3] a realization of this algebra was explicitly constructed by extending the dimensionality of the system and introducing an auxiliary variable [5]. The representation that the square-well states carry was not found, however, nor the algebraic analysis used for a specific application. On the other hand, in [4] the same system was used as a testing ground for the study of coherent states associated with anharmonic potentials. In this paper we find the ladder operators in terms of the physical variables of the system. We show below that the dynamical group is indeed

$SU(1,1)$, although our representation analysis differs from the work of [4]. This analysis may also be relevant in view of recent developments in the algebraic treatment of critical behaviour in nuclei, where the critical potential is approximated by an infinite square-well potential [6, 7].

Because of the anharmonic character and simplicity of the square-well potential, it represents an ideal system to study generalizations of the coherent state concept [4]. There are three ways to define a coherent state, (1) by means of the displacement operator acting on the vacuum state, (2) in terms of the eigenstates of the annihilation operator and (3) as minimum uncertainty states. The three definitions are equivalent for the harmonic oscillator [1], but differ in the case of anharmonic systems. In this work we analyse the coherent states according to the displacement operator formalism, also known as Perelomov's coherent states [8]. This definition differs from the ladder operator formalism leading to the Klauder–Glauber coherent states extensively studied in [4].

This paper is organized as follows. In section 2, we establish the raising and lowering operators directly from the eigenfunctions of the infinitely deep square-well potential with the factorization method [9–11]. The matrix elements of the functions $\cos(y)$ and $\sin(y)\,\mathrm{d}/\mathrm{d}y$ are obtained from these operators. Section 3 is devoted to the study of the infinitely deep symmetric square-well potential. In section 4 we discuss in detail the coherent states associated with this system, including the dispersion relations and the time dependence of the states. Finally, in section 5 we present our conclusions.

2. The ladder operators for the infinitely deep well

An infinitely deep square-well potential is defined as

$$V(x) = \begin{cases} 0 & \text{for } x \in [0, a] \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

The variable a corresponds to the width of the potential, which for simplicity will be chosen to be $a = 1$. With this selection for the length units, the solutions of the Schrödinger equation take the form

$$\Psi_n(y) = N \sin[(n+1)y] \qquad E_n = \hbar\omega(n+1)^2 \tag{2}$$

where $n = 0, 1, 2, \ldots,$ and

$$y = \pi x \qquad N = \sqrt{2} \qquad \omega = \frac{\hbar\pi^2}{2\mu} \tag{3}$$

where μ is the reduced mass of the particle. Note that the energy has been shifted in order to eliminate, from the description, the state with null energy.

Before constructing the ladder operators for this potential, we shall review the properties of the $SU(1,1)$ group, which has been found to be the dynamical group for the square-well potential [3, 4]. The $SU(1,1)$ group is non-compact [2]. Unlike the case of $SU(2)$ all its unitary irreducible representations are infinite-dimensional, and can be classified into three different kinds, the principal (continuous), discrete and supplementary series [8, 12]. In this work we shall be concerned only with the discrete series since the wavefunction (2) is associated with bound states. The $SU(1,1)$ generators $\hat{K}_{\pm,0}$ satisfy the following commutation relations:

$$[\hat{K}_+, \hat{K}_-] = -2\hat{K}_0 \qquad [\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm. \tag{4}$$

The discrete representations D_k^+ of this group have the standard form

$$\hat{K}_+|\kappa, m\rangle = \sqrt{(m+\kappa)(m-\kappa+1)}|\kappa, m+1\rangle \tag{5a}$$

$$\hat{K}_-|\kappa, m\rangle = \sqrt{(m-\kappa)(m+\kappa-1)}|\kappa, m-1\rangle \tag{5b}$$

$$\hat{K}_0|\kappa, m\rangle = m|\kappa, m\rangle \tag{5c}$$

where m can take the values

$$m = \kappa, \kappa+1, \ldots \tag{6}$$

for the Bargmann index $\kappa = 1/2, 1, 3/2, 2, \ldots$.

We now address the problem of finding raising and lowering operators for the wavefunctions (2), namely, we intend to search for differential operators $\hat{K}_\pm$ with the following property:

$$\hat{K}_\pm \Psi_n(y) = k_\pm \Psi_{n\pm 1}(y). \tag{7}$$

Specifically, we look for operators of the form

$$\hat{K}_\pm = A_\pm(y)\frac{\mathrm{d}}{\mathrm{d}y} + B_\pm(y) \tag{8}$$

where we stress that these operators depend only on the physical variable y. We start by finding the action of the differential operator $\frac{\mathrm{d}}{\mathrm{d}y}$ on the wavefunctions (2)

$$\frac{\mathrm{d}}{\mathrm{d}y}\Psi_n(y) = (n+1)N\cos[(n+1)y]. \tag{9}$$

Taking into account this result and defining the number operator $\hat{n}$ by the action

$$\hat{n}\Psi_n(y) = n\Psi_n(y) \tag{10}$$

we can readily prove the formula

$$\left[\cos(y)(\hat{n}+1) - \sin(y)\frac{\mathrm{d}}{\mathrm{d}y}\right]\frac{\hat{n}}{\hat{n}+1}\Psi_n(y) = n\Psi_{n-1}(y) \tag{11}$$

from which we can define the operator

$$\hat{K}_- = \left[\cos(y)(\hat{n}+1) - \sin(y)\frac{\mathrm{d}}{\mathrm{d}y}\right]\frac{\hat{n}}{\hat{n}+1} \tag{12}$$

with the following effect on the wavefunctions:

$$\hat{K}_-\Psi_n(y) = k_-\Psi_{n-1}(y) \tag{13}$$

where

$$k_- = n. \tag{14}$$

Note that as long as the operator $\hat{n}$ acts on the states $\Psi_n(y)$ its action in equations (11) and (12) is well defined. In this way k_- vanishes for the ground state. The function $\Psi_0(y)$ thus satisfies

$$\left[\cos(y) - \sin(y)\frac{\mathrm{d}}{\mathrm{d}y}\right]\Psi_0(y) = 0. \tag{15}$$

Here we stress the non-commutativity of the operators in (11). The operator $\hat{K}_-$ is determined by its action on the functions $\Psi_n(y)$. It is through this action that the number operator $\hat{n}$ is substituted by the number of quanta in accordance with (10).

We now proceed to find the corresponding raising operator. To this end we start with relation (9) and then use the following formula:

$$\left[\cos(y)(\hat{n}+1)+\sin(y)\frac{\mathrm{d}}{\mathrm{d}y}\right]\Psi_n(y)=(n+1)\Psi_{n+1}(y) \tag{16}$$

from which we can define the operator

$$\hat{K}_+=\cos(y)(\hat{n}+1)+\sin(y)\frac{\mathrm{d}}{\mathrm{d}y} \tag{17}$$

with the following effect on the wavefunctions:

$$\hat{K}_+\Psi_n(y)=k_+\Psi_{n+1}(y) \tag{18}$$

where

$$k_+=n+1. \tag{19}$$

Based on the results (13) and (18) we can calculate the commutator $[\hat{K}_+,\hat{K}_-]$:

$$[\hat{K}_+,\hat{K}_-]\Psi_n(y)=-2k_0\Psi_n(y) \tag{20}$$

where we have introduced the definition

$$k_0=n+\tfrac{1}{2}. \tag{21}$$

We can thus define the operator

$$\hat{K}_0=\hat{n}+\tfrac{1}{2}. \tag{22}$$

The operators $\hat{K}_{\pm,0}$ then satisfy the commutation relations

$$[\hat{K}_+,\hat{K}_-]=-2\hat{K}_0 \qquad [\hat{K}_0,\hat{K}_\pm]=\pm\hat{K}_\pm \tag{23}$$

which correspond to the $su(1,1)$ dynamical algebra for this potential.

Since the generators of the $SU(1,1)$ group satisfy relations (5), we need to establish the connection between the projection m and the quantum number n in order to recover (14), (19) and (21) from (5). This goal is accomplished by comparing the eigenvalues of the K_0 operator:

$$m=n+1/2. \tag{24}$$

With the substitution of (24) into (5) we obtain

$$\hat{K}_+|\kappa,n\rangle=\sqrt{(n+\kappa+1/2)(n-\kappa+3/2)}|n+1,\kappa\rangle \tag{25}$$

$$\hat{K}_-|\kappa,n\rangle=\sqrt{(n-\kappa+1/2)(n+\kappa-1/2)}|n-1,\kappa\rangle. \tag{26}$$

These equations reduce to (14) and (19), for $\kappa=1/2$.

We have thus established the dynamical group associated with the square-well potential, providing a realization for the generators in terms of the physical variables, which allows the determination of the representation series. We remark that these operators and their eigenvalues can be obtained naturally and directly from the wavefunctions (2) without introducing auxiliary variables as long as the definition (10) is considered. On the other hand, these results coincide with the standard representations (5) and differ from those given in [4], where identical eigenvalues $k_\pm=n$ are obtained for the ladder operators. Although the operators $\hat{K}_\pm$ are not symmetrical, the unitary representation (5) assures that $\hat{K}_\pm=\hat{K}_\mp^\dagger$ through the matrix elements

$$\langle\Psi_{n\pm1}|\hat{K}_\pm|\Psi_n\rangle=\left\langle\hat{K}_\pm^\dagger\Psi_{n\pm1}\middle|\Psi_n\right\rangle=\langle\hat{K}_\mp\Psi_{n\pm1}|\Psi_n\rangle. \tag{27}$$

The asymmetry seems to be necessary in order to reproduce the standard unitary representation (5). Symmetric expressions for the raising and lowering operators satisfying the $su(1,1)$

algebra, linear in the operator $\hat{n}$, are possible, but they do not lead to representation (5). In [3] such operators are obtained using the basis

$$f_n(y, \phi) = \Psi_n(y)\, e^{i(n+1)\phi} \tag{28}$$

where ϕ is an auxiliary variable introduced in order to extract the number of quanta n by means of the operator $i\partial/\partial\phi$ in the ladder operators. Because of the required normalization factor in (12), we have not followed this approach.

The corresponding Casimir operator can be written as

$$\hat{C} = \hat{K}_+\hat{K}_- - \hat{K}_0(\hat{K}_0 - 1) = \hat{K}_-\hat{K}_+ - \hat{K}_0(\hat{K}_0 + 1) \tag{29}$$

with eigenvalues

$$\hat{C}|\kappa, m\rangle = -\kappa(\kappa - 1)|\kappa, m\rangle. \tag{30}$$

For the value $\kappa = 1/2$ we obtain

$$\hat{C}\Psi_n(y) = \tfrac{1}{4}\Psi_n(y). \tag{31}$$

In addition, we note that for this representation the Hamiltonian acquires the simple form

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+ + 2\hat{K}_0) = \hbar\omega(\hat{K}_-\hat{K}_+) \tag{32}$$

while for the eigenfunctions

$$|\Psi_n\rangle = \frac{1}{n!}(\hat{K}_+)^n|\Psi_0\rangle. \tag{33}$$

It is interesting to note an alternative interpretation of the operators $\hat{K}_{\pm,0}$. We can define the operator

$$\hat{K}_+(\beta) = \beta\cos(y) + \sin(y)\frac{d}{dy} \tag{34}$$

in terms of which the eigenfunctions take the form

$$|\Psi_n\rangle = \frac{1}{n!}\prod_{\beta=1}^{n}\hat{K}_+(\beta)|\Psi_0\rangle. \tag{35}$$

This is the usual notation in supersymmetric quantum mechanics [13]. Note that in (35) a pre-established order in the action of the operators must be applied in order to reproduce the function $|\Psi_n\rangle$. In similar form we can define the lowering operator

$$\hat{K}_-(\gamma) = \frac{(\gamma - 1)}{\gamma}\left[\gamma\cos(y) - \sin(y)\frac{d}{dy}\right] \tag{36}$$

with the commutation relation

$$[\hat{K}_+(\beta), \hat{K}_-(\gamma)] = -\frac{(\beta+\gamma)(\gamma-1)}{\gamma}\sin^2(y). \tag{37}$$

One of the advantages of the algebraic method consists in simplifying the calculation of matrix elements of operators. This is achieved by expressing them in terms of the generators of the group. In our case, from the realization of the ladder operators $\hat{K}_\pm$ we obtain

$$\cos(y) = \frac{1}{2}\left(\hat{K}_+\frac{1}{\hat{n}+1} + \hat{K}_-\frac{1}{\hat{n}}\right) \tag{38}$$

and

$$\sin(y)\frac{d}{dy} = \frac{1}{2}\left(\hat{K}_+ - \hat{K}_-\frac{\hat{n}+1}{\hat{n}}\right). \tag{39}$$

In spite of the peculiar form of equations (38) and (39), these expressions are very useful. The matrix elements of these two functions can be readily obtained from equations (13) and (18) as

$$\langle m|\cos(y)|n\rangle = \tfrac{1}{2}(\delta_{m,n+1} + \delta_{m,n-1}) \tag{40}$$

and

$$\langle m|\sin(y)\frac{\mathrm{d}}{\mathrm{d}y}|n\rangle = \langle m|\frac{1}{\pi}\sin(\pi x)\frac{\mathrm{d}}{\mathrm{d}x}|n\rangle = \frac{n+1}{2}(\delta_{m,n+1} - \delta_{m,n-1}). \tag{41}$$

3. Infinitely deep symmetric well potential

For the sake of completeness and in order to study wavefunctions with good parity, we now consider that the potential has the form:

$$V(x) = \begin{cases} 0 & \text{for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \infty & \text{otherwise} \end{cases} \tag{42}$$

with the same selection for the length unit $a = 1$. The solution of the Schrödinger equation for this potential is then given by

$$\Psi_n(y) = \begin{cases} N\sin[(n+1)y] & \text{when } n = \text{odd} \\ N\cos[(n+1)y] & \text{when } n = \text{even} \end{cases} \tag{43}$$

where $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and again we have shifted the energies, so that $n = 1, 2, 3, \ldots$.

Since the approach to find the ladder operators is similar to the one previously presented, we only summarize the results. The realization of the dynamical algebra turns out to be

$$\hat{L}_- = \left[\sin(y)(\hat{n}+1) + \cos(y)\frac{\mathrm{d}}{\mathrm{d}y}\right]\frac{\hat{n}}{\hat{n}+1} \tag{44a}$$

$$\hat{L}_+ = \left[\sin(y)(\hat{n}+1) - \cos(y)\frac{\mathrm{d}}{\mathrm{d}y}\right] \tag{44b}$$

$$\hat{L}_0 = \hat{n} + \frac{1}{2} \tag{44c}$$

with the following action on the wavefunctions:

$$\hat{L}_-\Psi_n(y) = l_-\Psi_{n-1}(y) = (-1)^{n+1}n\Psi_{n-1}(y) \tag{45a}$$

$$\hat{L}_+\Psi_n(y) = l_+\Psi_{n+1}(y) = (-1)^{n}(n+1)\Psi_{n+1}(y) \tag{45b}$$

$$\hat{L}_0\Psi_n(y) = l_0\Psi_n(y) = \left(n + \tfrac{1}{2}\right)\Psi_n(y). \tag{45c}$$

The operators $\hat{L}_{\pm,0}$ satisfy the commutation relations

$$[\hat{L}_+, \hat{L}_-] = -2\hat{L}_0 \qquad [\hat{L}_0, \hat{L}_\pm] = \pm\hat{L}_\pm \tag{46}$$

which also correspond to the $SU(1,1)$ dynamical group, as expected. It is found, however, that in this case the ladder operators $\hat{L}_\pm$ change the parity of the wavefunctions. Using the eigenvalues of the generators we can check that the eigenvalue of the Casimir operator is again $1/4$, corresponding to $\kappa = 1/2$. We also see that $m = n + 1/2$, and consequently equations (45) should be obtained from (25) and (26). This is the case only if a phase in the definition of the representation is introduced in the following way:

$$\hat{K}_+|\kappa, n\rangle = (-1)^n\sqrt{(n+\kappa+1/2)(n-\kappa+3/2)}|n+1, \kappa\rangle \tag{47a}$$

$$\hat{K}_-|\kappa, n\rangle = (-1)^{n+1}\sqrt{(n-\kappa+1/2)(n+\kappa-1/2)}|n-1, \kappa\rangle \tag{47b}$$

which reduce to (41), for $\kappa = 1/2$.

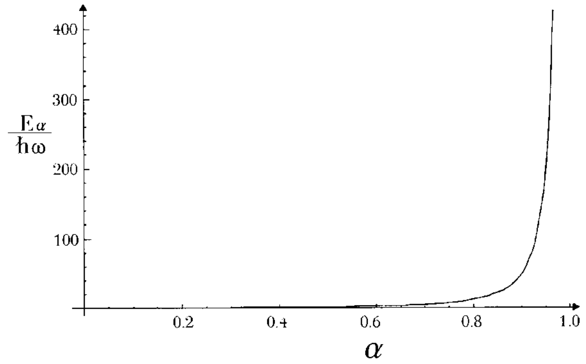

Figure 1. Expectation value of the energy as a function of the parameter α.

4. Coherent state analysis

Recently, coherent states for the infinite square-well potential were discussed using the ladder operator formalism [4], abstractly involving the dynamical algebra $su(1,1)$ [14]. In this work we shall analyse the same problem but in the framework of the displacement operator method. We shall see that this approach leads to very simple forms for these states. Perelomov's coherent states are given by [8]

$$|\kappa,\alpha\rangle = D(\alpha)|\kappa,0\rangle = N_{\kappa,\alpha}\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(2\kappa+n)}{\Gamma(2\kappa)n!}}\alpha^n|\kappa,n\rangle \tag{48}$$

where $|\alpha| < 1$, N_α is the normalization constant and $D(\alpha) = \exp(\alpha\hat{K}_+ - \alpha^*\hat{K}_-)$. For our system $\kappa = 1/2$, according to the discussion leading to representation (5), and consequently equation (48) reduces to

$$\left|\frac{1}{2},\alpha\right\rangle = N_{\frac{1}{2},\alpha}\sum_{n=0}^{\infty}\alpha^n\left|\frac{1}{2},n\right\rangle = N_{\frac{1}{2},\alpha}\sqrt{2}\sum_{n=0}^{\infty}\alpha^n\sin[(n+1)y] \tag{49}$$

with

$$N_{\frac{1}{2},\alpha} = \sqrt{1-|\alpha|^2}. \tag{50}$$

The resolution of the identity has been proved by Vourdas and Wünsche [15] in terms of contour integrals of $SU(1,1)$ coherent states.

The sum involved in (49) can be obtained in closed form

$$\left\langle x\left|\frac{1}{2},\alpha\right.\right\rangle = \frac{\sqrt{2}\sqrt{1-|\alpha|^2}\sin(\pi x)}{1+|\alpha|^2-2\alpha\cos(\pi x)}. \tag{51}$$

Using the coherent states (51) we can also evaluate the expectation value of the energy, which turns out to be

$$E_\alpha = \left\langle\frac{1}{2},\alpha\right|\hat{H}\left|\frac{1}{2},\alpha\right\rangle = \hbar\omega N_\alpha^2\sum_{n=0}^{\infty}\alpha^{2n}(n+1)^2 = \hbar\omega\frac{(1+|\alpha|^2)}{(1-|\alpha|^2)^2}. \tag{52}$$

The relation between α and the energy is displayed in figure 1, which closely resembles the shape of the original square-well potential.

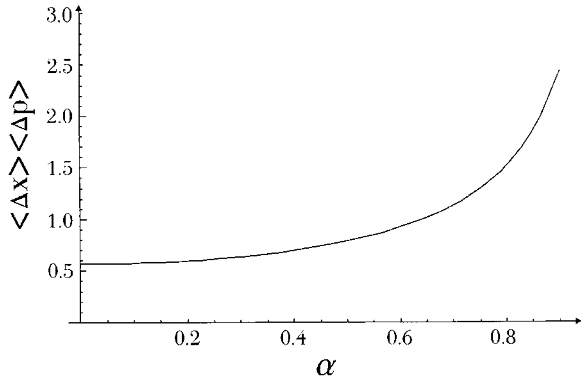

Figure 2. Uncertainty product $\langle\Delta x\rangle\langle\Delta p\rangle$ as a function of α.

The coherent states can also be defined in terms of the minimum uncertainty states. The generalization of this approach and its relation to the displacement method was studied by Trifonov [16]. Although we are not concerned in this work with the generalized uncertainty relation, it is interesting to calculate $\langle\Delta x\rangle\langle\Delta p\rangle$. The uncertainties $\langle\Delta x\rangle$ and $\langle\Delta p\rangle$ are defined as

$$\langle\Delta x\rangle = \sqrt{\left\langle\tfrac{1}{2},\alpha\middle|x^2\middle|\tfrac{1}{2},\alpha\right\rangle - \left\langle\tfrac{1}{2},\alpha\middle|x\middle|\tfrac{1}{2},\alpha\right\rangle^2} \tag{53a}$$

$$\langle\Delta p\rangle = \sqrt{\left\langle\tfrac{1}{2},\alpha\middle|p^2\middle|\tfrac{1}{2},\alpha\right\rangle - \left\langle\tfrac{1}{2},\alpha\middle|p\middle|\tfrac{1}{2},\alpha\right\rangle^2}. \tag{53b}$$

Using the explicit form (51) for the coherent states, it is possible to obtain analytic expressions for the expectation values of the momentum $p = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ and p^2:

$$\left\langle\frac{1}{2},\alpha\middle| p \middle|\frac{1}{2},\alpha\right\rangle = 0 \qquad \left\langle\frac{1}{2},\alpha\middle| p^2 \middle|\frac{1}{2},\alpha\right\rangle = \hbar^2\frac{\pi^2(1+\alpha^2)}{(\alpha^2-1)} \tag{54}$$

from which we obtain for the uncertainty

$$\langle\Delta p\rangle = \hbar\pi\sqrt{\frac{(1+\alpha^2)}{(\alpha^2-1)}}. \tag{55}$$

Unfortunately, even for this simple system the uncertainty for x cannot be calculated in closed form. Hence we have proceeded to obtain the behaviour numerically. While $\langle\Delta x\rangle$ tends to vanish for $\alpha \to 1$, the uncertainty in the momentum increases so that the uncertainty relation is satisfied. In figure 2 we display the product $\langle\Delta x\rangle\langle\Delta p\rangle$ as a function of α. We find that the uncertainty relation

$$\langle\Delta x\rangle\langle\Delta p\rangle \geqslant \frac{\hbar}{2} \tag{56}$$

has a minimum corresponding to $\alpha = 0$, for which it takes the value

$$0.5678 \geqslant \tfrac{1}{2} \tag{57}$$

and increases as $\alpha \to 1$, as shown in the figure. This result was expected since the displacement operator approach to obtaining the coherent states is not compatible with the minimum uncertainty states satisfying the equality in (56).

We next study the temporal features of the coherent states. Since the coherent states are given in terms of a superposition of eigenstates of a time independent Hamiltonian, their time evolution is given by

$$\left|\tfrac{1}{2}, \alpha, t\right\rangle = \mathrm{e}^{-\mathrm{i}\frac{\hat{H}t}{\hbar}}\left|\tfrac{1}{2}, \alpha\right\rangle \tag{58}$$

and explicitly

$$\left|\frac{1}{2}, \alpha, t\right\rangle = N_{\frac{1}{2},\alpha} \sum_{n=0}^{\infty} \alpha^n \, \mathrm{e}^{-\mathrm{i}\frac{\hat{E}_n t}{\hbar}} \left|\frac{1}{2}, n\right\rangle = N_{\frac{1}{2},\alpha} \sum_{n=0}^{\infty} \alpha^n \, \mathrm{e}^{-\mathrm{i}\frac{\hbar\omega(n+1)^2 t}{\hbar}} \left|\frac{1}{2}, n\right\rangle. \tag{59}$$

In general the time evolution of the states does not preserve its initial form. How close the original states (with $t = 0$) are reproduced depends on the value of α. Only for small values of α does the evolution of the states maintain the initial form for a long time. As α increases, the states deform appreciably with time, which is in accordance with the results for the uncertainty relation displayed in figure 2.

5. Concluding remarks

In this work we have established a realization in terms of the physical variables of the raising and lowering operators for the infinitely deep square-well potential by means of the factorization method. It is shown that the $SU(1, 1)$ group is the dynamical group for the bound states of both the symmetric and asymmetric systems. A representation analysis was presented, where it was found that the system is described by the discrete series associated with the Bargmann index $\kappa = 1/2$. The matrix elements of the ladder operators as well as the eigenvalue of the Casimir operator differ from those obtained in [3]. While we have derived a realization only in terms of the physical variable y, in [3] an auxiliary variable was introduced. We have obtained a realization of the $su(1, 1)$ algebra leading to the standard representation (5). However, in order to achieve this goal it was necessary to introduce a normalization factor which breaks the symmetry of the operators. In spite of the asymmetry the ladder operators $\hat{K}_\pm$ continue to be adjoint partners, a consequence of the unitary representation (5).

Following the definition in terms of the exponentiation of ladder operators, we have obtained the Perelomov coherent states associated with this problem. Analytic results can be found for most observables. An analysis of the dispersion relation as well as a study of the time evolution of the states was carried out. These results represent a simple example of coherent states for anharmonic potentials where one can gauge their properties as compared with the harmonic oscillator ones [17]. In a different context the paper may provide a starting point in the analysis of physically relevant solutions of the Bohr–Mottelson Hamiltonian designed to describe critical behaviour in the quantum shape transition of atomic nuclei [6, 7].

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